THE RESTRICTED THREE-BODY PROBLEM: FORMULATION, DYNAMICAL STRUCTURES, AND APPLICATIONS TO LOW-ENERGY SPACE MISSION DESIGN

O PROBLEMA RESTRITO DE TRÊS CORPOS: FORMULAÇÃO, ESTRUTURAS DINÂMICAS E APLICAÇÕES AO PROJETO DE MISSÕES ESPACIAIS DE BAIXA ENERGIA

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ABSTRACT

The preliminary design of interplanetary and lunar missions often employs the patched conics method, based on two-body problem solutions. However, mission parameters may require more complex gravitational models, such as the Restricted Three-Body Problem (RTBP), which has been extensively used in modern space mission design. This work presents a detailed formulation of the RTBP, including its dimensionless form, the effective potential, the Jacobi integral, and the concept of Hill's regions. The equilibrium points (Lagrange points) and their respective energy levels are analyzed, along with the five basic energy cases that define the possible regions of motion for a spacecraft of negligible mass. The results highlight the importance of the RTBP in low-energy trajectory design, offering a foundation for missions requiring fuel-efficient transfers and ballistic captures.

Keywords: Restricted Three-Body Problem. Jacobi Integral. Lagrange Points. Hill's Regions. Low-Energy Trajectories.

RESUMO

O projeto preliminar de missões interplanetárias e lunares frequentemente utiliza o método das cônicas conjugadas, baseado em soluções do problema de dois corpos. No entanto, parâmetros de missão podem exigir modelos gravitacionais mais complexos, como o Problema Restrito de Três Corpos (PRTC), amplamente empregado no projeto de missões espaciais modernas. Este trabalho apresenta uma formulação detalhada do PRTC, incluindo sua forma adimensional, o potencial efetivo, a integral de Jacobi e o conceito de regiões de Hill. Os pontos de equilíbrio (pontos de Lagrange) e seus respectivos níveis de energia são analisados, juntamente com os cinco casos básicos de energia que definem as regiões possíveis de movimento para uma espaçonave de massa desprezível. Os resultados destacam a importância do PRTC no projeto de trajetórias de baixa energia, oferecendo uma base para missões que requerem transferências eficientes em termos de combustível e capturas balísticas.

Palavras-chave: Problema Restrito de Três Corpos. Integral de Jacobi. Pontos de Lagrange. Regiões de Hill. Trajetórias de Baixa Energia.

1 INTRODUÇÃO

The preliminary design of interplanetary and lunar space missions represents a complex challenge within aerospace engineering and celestial mechanics. Traditionally, mission planning is conducted using the patched conics method, which describes the spacecraft's trajectory through successive arcs of conic solutions of the Kepler Problem, considering only the gravitational interaction between two bodies (COSTA & SOUSA-SILVA). Although this approach is effective for initial parameter estimation, it presents significant limitations when dealing with maneuvers that require low fuel consumption or ballistic captures around celestial bodies.

A milestone in overcoming these limitations occurred in 1991 when E. Belbruno proposed a low-thrust transfer technique that was crucial for the rescue of the Japanese spacecraft *Hiten*. By considering solar gravitational perturbations within the Two-Body Problem framework, a ballistic lunar capture was achieved with a significant reduction in propellant usage compared to the originally planned orbital maneuver (BELBRUNO; CAPDEVILLE). This achievement demonstrated that incorporating perturbative forces can open new possibilities for low-energy trajectories.

However, more realistic models, such as n-body problems, generally lack closed-form analytical solutions and exhibit chaotic behavior, making trajectory planning a challenging task (VASILE). In this context, the Restricted Three-Body Problem (RTBP) emerges as a highly relevant intermediate model. It describes the dynamics of a negligible-mass body under the gravitational influence of two massive bodies orbiting each other, allowing the identification of fundamental structures such as Lagrange points, periodic orbits, and invariant manifolds. These structures are essential for determining new types of low-energy trajectories (VASILE).

Thus, this work aims to present a detailed formulation of the Restricted Three-Body Problem, addressing its non-dimensionalization, equations of motion, the Jacobi integral, Hill's regions, and the classification of equilibrium points. Understanding these concepts provides the theoretical foundation necessary for the design of modern space missions that seek efficiency and innovation in orbital transfers.

2 LITERATURE REVIEW

The modeling of spacecraft orbital dynamics has evolved significantly in recent decades, moving from simplified two-body approaches to more complex models that incorporate multiple gravitational influences. This section reviews the theoretical foundations of the Restricted Three-Body Problem, highlighting its mathematical properties and implications for space flight mechanics (COSTA & SOUSA-SILVA).

2.1 From the Two-Body Problem to the Restricted Three-Body Problem

The Two-Body Problem, solved by Kepler and Newton, assumes that only two bodies interact gravitationally, resulting in conic orbits. Although useful for basic orbital maneuvers, it does not account for perturbations from other celestial bodies. The Restricted Three-Body Problem (RTBP) emerges as a natural extension, introducing a third body of negligible mass whose motion is influenced by two massive primaries, without altering their motion (SZEBEHELY).

According to Vasile, this formulation is particularly relevant for interplanetary missions, as it allows modeling the combined influence of two gravitating bodies, such as Earth and the Moon, or the Sun and Earth. Although the RTBP lacks a closed-form analytical solution due to the system's non-integrability, it provides fundamental geometric structures that can be exploited for low-energy trajectory construction.

2.2 Non-Dimensionalization and Mass Parameter

To simplify mathematical analysis, the RTBP is often expressed in a normalized unit system. As described by Koon et al., the unit of mass is defined as the sum of the primaries' masses $(m_1 + m_2)$, the unit of length is the constant distance between them, and the unit of time is adjusted so that the orbital period of the primaries is 2π. In this system, the gravitational constant becomes G = 1, and the system's dynamics depend solely on the mass parameter μ, given by:

$$\mu = \frac{m_2}{m_1 + m_2}$$

For the Earth-Moon system, for example, μ is approximately 0.01215, reflecting the large mass difference between the two primaries (SZEBEHELY).

2.3 Equations of Motion in the Rotating Frame

The classical RTBP formulation adopts a non-inertial reference frame that rotates with the same angular velocity as the primaries, fixing them on the x-axis. In this system, the equations of motion for the third body are given by:

$$\ddot{x} - 2\dot{y} = \Omega_x$$
$$\ddot{y} + 2\dot{x} = \Omega_y$$

where Ω_x and Ω_y are partial derivatives of the effective potential Ω, which incorporates both gravitational attraction and centrifugal force (SZEBEHELY). The effective potential is defined as:

$$\Omega = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2}$$

with r_1 and r_2 representing the distances from the third body to P_1 and P_2, respectively.

2.4 The Jacobi Integral and Energy Conservation

A fundamental property of the RTBP is the existence of a time-independent integral of motion, known as the Jacobi Integral. Derived from the Hamiltonian nature of the system, this integral represents the energy of the third body in the rotating frame. The form commonly used in celestial mechanics is:

$$C(x, y, \dot{x}, \dot{y}) = -\left(\dot{x^2} + \dot{y^2}\right) - 2\Omega(x, y, z)$$

As highlighted by Koon et al., this constant relates the kinetic energy of the body to the effective potential. The lower the value of C, the higher the energy of the system. The absence of other integrals of motion gives the system a chaotic character, but the Jacobi Integral allows classification of possible motion types based on energy levels.

2.5 Hill's Regions and Zero-Velocity Curves

The energy surface $\mathcal{M}(\mu, C)$ defines the regions in phase space where the third body's motion is permitted. The projection of this surface onto the position plane (x, y)

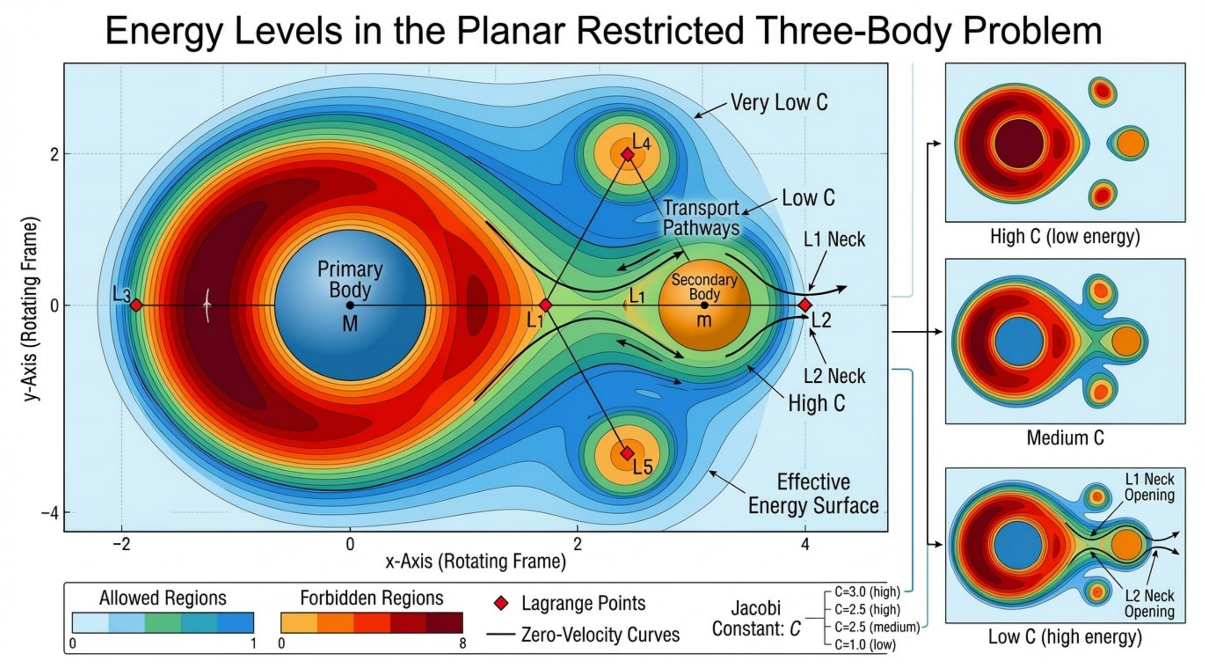
results in the so-called Hill's Region, which delimits the accessible areas for the body at a given energy level (KOON et al.).

The boundary of this region is defined by the zero-velocity curves, where kinetic energy vanishes:

$$\frac{1}{2}v^2(x,y) = C - \Omega(x,y) = 0$$

The side of the curve where $C - \Omega < 0$ represents the forbidden domain, while the opposite side indicates zones of possible motion. These curves vary with energy C, giving rise to five basic configurations, each associated with one of the Lagrange points (Picture 1).

Picture 1 – Energy Levels in the RTBP



Source: Own authorship.

2.6 *Lagrange Points and Classification of Energy Cases*

Lagrange points are equilibrium solutions of the RTBP, obtained by setting the derivatives of the effective potential to zero. There are five notable points: three collinear (L_1, L_2, L_3) located on the axis joining the primaries, and two triangular points (L_4, L_5) that form equilateral triangles with the primaries (SZEBEHELY).

The coordinates of the triangular points are given by $x = \mu - 1/2, y = \pm\sqrt{3}/2$, while the collinear points are determined by solving fifth-degree algebraic equations. The energies associated with these points follow the relationship:

$$E_5 = E_4 > E_3 > E_2 > E_1$$

Based on energy levels, Koon et al. classify five energy cases that determine the topology of Hill's regions:

- **Case 1**: Energy below E_1 — the domains around each primary are isolated.
- **Case 2**: Energy between E_1 and E_2 — a gateway opens at L_1, allowing transfer between the primaries' domains.
- **Case 3**: Energy between E_2 and E_3 — a gateway opens at L_2, connecting the m_2 domain to the exterior region.
- **Case 4**: Energy between E_3 and E_4 — a gateway at L_3 directly connects m_1 to the exterior.
- **Case 5**: Energy above E_4 — no forbidden domains exist, allowing motion throughout the entire xy-plane.

These cases are fundamental for trajectory design, as they indicate under which energy conditions ballistic capture or gravity-assisted escape maneuvers are possible.

2.7 Properties of the Effective Potential and Dynamical Implications

The effective potential Ω plays a central role in determining regions of motion and locating equilibrium points. Several properties can be extracted from its analysis, as systematized by Szebehely:

- The minimum value of Ω is 3/2, occurring at the triangular points L_4 and L_5.
- $\Omega \to \infty$ as the distance from the barycenter increases, indicating the dominant influence of centrifugal force in distant regions.
- The potential is symmetric with respect to the x-axis, reflecting the symmetry of the problem.
- The collinear points are saddle points, while the triangular points are local minima.

- Zero-velocity curves are orthogonal to the potential gradient at regular points.

These properties enable the classification of the system's dynamics and are widely used in the construction of low-energy trajectories and the study of orbital stability in binary systems.

2.8 Poincaré Sections

The analysis of dynamical systems, particularly those with chaotic behavior such as the Restricted Three-Body Problem (RTBP), often requires tools that reduce the dimensionality of the problem while preserving essential information about the system's structure. One of the most powerful and widely used techniques for this purpose is the Poincaré section, introduced by Henri Poincaré in his foundational work on celestial mechanics.

A Poincaré section is a method of reducing a continuous dynamical system to a discrete map by intersecting the system's flow with a lower-dimensional surface (the section) in phase space. Instead of analyzing a continuous trajectory over time, one records only the points where the trajectory crosses a predefined surface, typically a plane, with a consistent direction (e.g., from one side to the other). This transformation simplifies the visualization and analysis of complex motion, revealing underlying structures such as invariant tori, periodic orbits, and chaotic regions (SZEBEHELY; COSTA & SOUSA-SILVA).

In the context of the RTBP, Poincaré sections are commonly constructed by fixing a specific value of the Jacobi integral C and selecting a surface of section, such as the plane $x = \mu - 1$ (the vertical line through L_2) or $y = 0$, depending on the region of interest. For a given energy level, the motion of the third body is confined to a three-dimensional energy manifold. By taking a two-dimensional surface within this manifold, the continuous flow is reduced to a two-dimensional map, allowing for a detailed investigation of the system's dynamics (KOON et al.).

The resulting Poincaré map exhibits distinct features that characterize the nature of the motion:

- **Quasi-periodic motion** appears as closed curves (invariant tori) in the Poincaré section, indicating regular, stable trajectories.

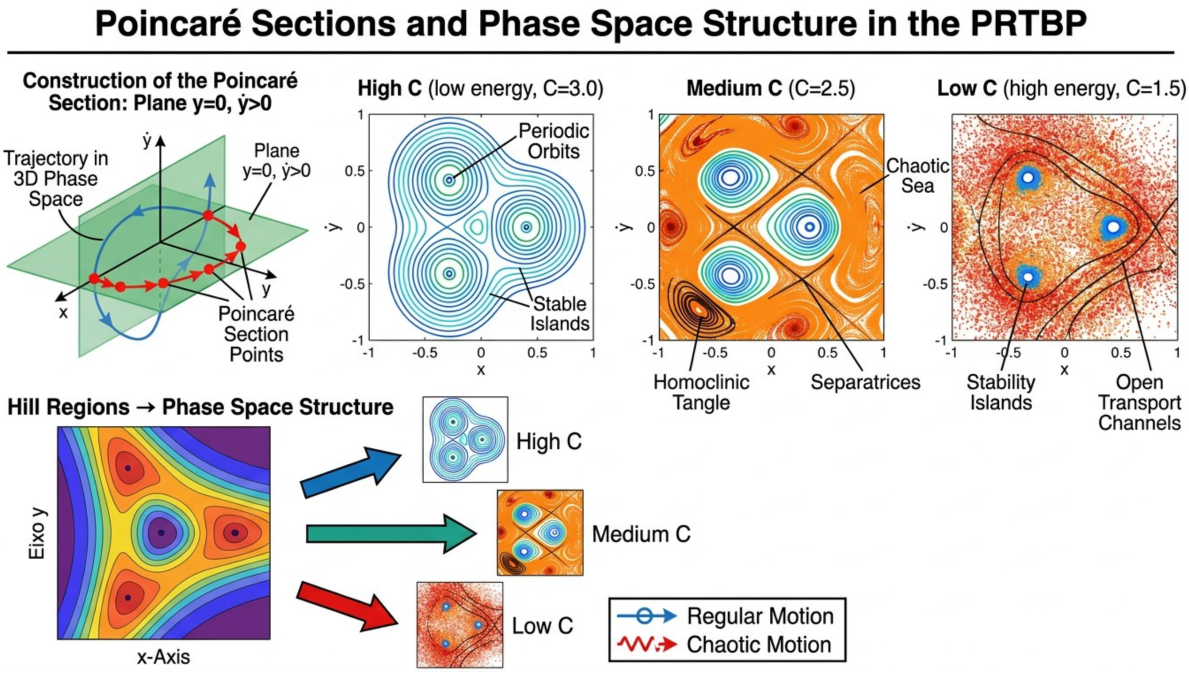
- **Periodic orbits** correspond to isolated fixed points or small chains of points in the map.
- **Chaotic motion** manifests as scattered, irregular points that fill regions of the section, indicating sensitivity to initial conditions and complex, unpredictable behavior.

Costa and Sousa-Silva (2023) emphasize the importance of Poincaré sections in the study of the RTBP, particularly for identifying regions of stability and instability around Lagrange points. Their work demonstrates how Poincaré sections can be used to visualize the transitions between different energy cases, revealing the opening of gateways as energy increases and the subsequent emergence of chaotic layers that facilitate low-energy transfers.

The application of Poincaré sections extends beyond theoretical analysis. In mission design, these maps are employed to identify initial conditions that lead to desired trajectories, such as transfers between libration point orbits or ballistic captures. By analyzing the structure of the Poincaré map, engineers can locate stable manifolds that guide spacecraft naturally toward target orbits with minimal fuel consumption (VASILE).

Thus, Poincaré sections serve as an indispensable tool for both qualitative and quantitative analysis of the RTBP, bridging the gap between theoretical dynamical systems theory and practical astrodynamics applications.

Picture 2 – Poincaré Sections and RTBP Phase Space Structure



Source: Own authorship.

3 FINAL REMARKS

The Restricted Three-Body Problem (RTBP) constitutes a fundamental theoretical framework for modern astrodynamics, providing essential insights into the design of low-energy space missions. This work presented a comprehensive formulation of the RTBP, addressing its mathematical foundations, including the non-dimensionalization of the system, the equations of motion in the rotating frame, and the definition of the effective potential. Special attention was given to the Jacobi Integral as the sole conserved quantity, which plays a crucial role in delimiting the regions of possible motion through the zero-velocity curves and Hill's regions.

The analysis of the five Lagrange points revealed their distinct dynamical nature: the collinear points (L_1, L_2, L_3) act as saddle points, while the triangular points (L_4, L_5) correspond to local minima of the effective potential. The energy hierarchy associated with these points directly determines the topological configurations of Hill's regions, which transition from isolated domains around each primary to a fully accessible plane as energy increases. These five energy cases provide a roadmap for trajectory design, indicating the specific energy thresholds that enable ballistic capture, gravitational assists, and low-energy transfers.

Furthermore, the use of Poincaré sections was discussed as an essential tool for reducing the dimensionality of the RTBP and revealing the underlying dynamical structures. By intersecting the flow with a suitable surface, Poincaré maps allow the visualization of regular and chaotic regions, invariant tori, periodic orbits, and the transitions between distinct energy regimes. As demonstrated by Costa and Sousa-Silva (2023), these sections are particularly valuable for identifying stability zones around Lagrange points and characterizing the chaotic layers that facilitate low-energy transfers. The integration of Poincaré section analysis into mission design enables the identification of initial conditions that leverage natural dynamical pathways, significantly reducing fuel consumption.

The theoretical concepts explored here have direct practical applications. The use of invariant manifolds associated with Lagrange points, for instance, has enabled groundbreaking missions such as the rescue of *Hiten*, as well as the design of trajectories for spacecraft like the Solar and Heliospheric Observatory (SOHO) and the James Webb Space Telescope, which utilize halo orbits around L_1 and L_2.

Furthermore, the RTBP serves as a foundational model for more complex problems, including the Elliptic Restricted Three-Body Problem, the Bi-Circular Model, and full ephemeris models used in operational mission design.

It is important to note that while the RTBP offers a robust theoretical framework, its limitations must be acknowledged. The assumption of circular orbits for the primaries and the neglect of the third body's gravitational influence restrict its applicability to certain mission phases. Nevertheless, the RTBP remains an indispensable tool for preliminary mission analysis, providing initial estimates and revealing dynamical structures that can later be refined with higher-fidelity models.

In conclusion, the understanding of the Restricted Three-Body Problem and its associated dynamical structures is essential for the development of efficient and innovative space missions. The integration of these concepts into the preliminary design phase allows for significant fuel savings and enables trajectories that would be impossible under the two-body approximation. Future work may extend this analysis to more complex models, incorporating elliptic motions, solar radiation pressure, and other perturbations, as well as exploring the application of machine learning techniques for trajectory optimization within the RTBP framework.

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REFERENCES

Belbruno, E. (2004). *Capture dynamics and chaotic motions in celestial mechanics: With applications to the construction of low energy transfers*. Princeton University Press.

Capdeville, Y. (2002). *Low-energy transfers and stability of the Hiten mission*. In *Proceedings of the International Symposium on Space Flight Dynamics* (pp. 123–130). CNES.

Costa, J. G. da, & Sousa-Silva, P. A. de. (2023). Seções de Poincaré do Problema Restrito de Três Corpos. *Proceeding Series of the Brazilian Society of Computational and Applied Mathematics*, *10*(1). https://doi.org/10.5540/03.2023.010.01.0460

Koon, W. S., Lo, M. W., Marsden, J. E., & Ross, S. D. (2011). *Dynamical systems, the three-body problem and space mission design*. Springer.

Ross, S. D., Koon, W. S., Lo, M. W., & Marsden, J. E. (2022). *Low-energy transfers and invariant manifolds in the restricted three-body problem*. *Celestial Mechanics and Dynamical Astronomy*, 134(2), 1–25. https://doi.org/10.1007/s10569-022-10067-3

Szebehely, V. (1967). *Theory of orbits: The restricted problem of three bodies*. Academic Press.

Vasile, M. (2019). *Low-thrust and low-energy trajectories*. In *Encyclopedia of aerospace engineering* (pp. 1–15). John Wiley & Sons. https://doi.org/10.1002/9780470686652.eae1028