QUANTITATIVE ANALYSIS OF CHAOS IN THE EARTH-MOON SYSTEM: CORRELATION BETWEEN LARGEST LYAPUNOV EXPONENT MAPS AND ESCAPE BASINS IN THE RESTRICTED THREE-BODY PROBLEM

ANÁLISE QUANTITATIVA DO CAOS NO SISTEMA TERRA-LUA: CORRELAÇÃO ENTRE MAPAS DO MAIOR EXPOENTE DE LYAPUNOV E BACIAS DE ESCAPE NO PROBLEMA RESTRITO DE TRÊS CORPOS

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Júlia Gomes da Costa [1]
Denilson Paulo Souza dos Santos [2]

[1] Graduanda em Engenharia Aeronáutica. Universidade Estadual Paulista "Júlio de Mesquita Filho" – UNESP, Campus de São João da Boa Vista.
[2] Professor Associado, Universidade Estadual Paulista "Júlio de Mesquita Filho" – UNESP, Campus de São João da Boa Vista.

ABSTRACT

The design of low-energy space missions requires a deep understanding of transport mechanisms in the Earth-Moon system, often modeled by the Planar Circular Restricted Three-Body Problem (PCRTBP). While escape basins provide a qualitative classification of trajectory outcomes, they do not provide a quantitative measure of orbital instability. This work presents a quantitative analysis of chaos by correlating the Largest Lyapunov Exponent (LLE) with the basin structures. We implemented the periodic renormalization algorithm to compute the LLE for a grid of initial conditions in the lunar region. Maps of the LLE were generated for three distinct Jacobi constants (C=3.08, 3.15, 3.2004), corresponding to high, intermediate, and low energy levels. The results reveal a direct correlation between the fractal basin boundaries and regions of high LLE. The intermediate energy regime exhibits the most complex dynamical structure, with the highest LLE values, confirming the high sensitivity to initial conditions and the chaotic nature of the transition between escape and confinement. The high-energy regime (C=3.08) shows widespread positive LLE, while the low-energy regime (C=3.2004) is predominantly regular. This quantitative approach provides crucial information for trajectory design, identifying stable regions and unpredictability zones.

Keywords: Restricted Three-Body Problem. Largest Lyapunov Exponent. Escape Basins. Chaos Quantification. Earth-Moon System.

RESUMO

O projeto de missões espaciais de baixa energia demanda uma compreensão profunda dos mecanismos de transporte no sistema Terra-Lua, frequentemente modelado pelo Problema Restrito de Três Corpos Circular Planar (PRTCCP). Embora as bacias de escape forneçam uma classificação qualitativa dos resultados da trajetória, elas não oferecem uma medida quantitativa da instabilidade orbital. Este trabalho apresenta uma análise quantitativa do caos, correlacionando o Maior Expoente de Lyapunov (LLE) com as estruturas das bacias. Implementamos o algoritmo de renormalização periódica para calcular o LLE para uma grade de condições iniciais na região lunar. Mapas do LLE foram gerados para três constantes de Jacobi distintas (C=3.08, 3.15, 3.2004), correspondendo a níveis de energia alto, intermediário e baixo. Os resultados revelam uma correlação direta entre as fronteiras fractais das bacias e as regiões de alto LLE. O regime de energia intermediária exibe a estrutura dinâmica mais complexa, com os maiores valores de LLE, confirmando a alta sensibilidade às condições iniciais e a natureza caótica da transição entre escape e confinamento. O regime de alta energia (C=3,08) mostra LLE positivos generalizados, enquanto o regime de baixa energia (C=3,2004) é predominantemente regular. Esta abordagem quantitativa fornece informações cruciais para o projeto de trajetórias, identificando regiões estáveis e zonas de imprevisibilidade.

Palavras-chave: Problema Restrito de Três Corpos. Maior Expoente de Lyapunov. Bacias de Escape. Quantificação do Caos. Sistema Terra-Lua.

1 INTRODUCTION

The preliminary design of interplanetary and lunar space missions often employs the patched conic method, based on solutions to the two-body problem. However, for missions aiming for maximum energy efficiency, such as those utilizing low-thrust transfers and ballistic capture, it is necessary to resort to more complex gravitational models. In this context, the Restricted Three-Body Problem (RTBP) has been widely employed (KOON et al., 2006, COSTA and SANTOS, 2026).

In the RTBP model, a particle of negligible mass moves under the gravitational influence of two primary bodies (e.g., Earth and Moon) that describe circular orbits. One of the most powerful tools for analyzing the resulting dynamics is escape basins. These consist of maps that classify the final fate (collision, escape to the Earth's domain, escape to the exterior domain, or bounded orbit) of a grid of initial conditions (ASSIS; TERRA, 2014). The complexity of these basins, with their fractal boundaries, serves as a qualitative indicator of the system's chaotic nature, revealing a high sensitivity to initial conditions.

However, the qualitative analysis of the basins has a limitation: it does not quantify the rate of instability or chaos. A trajectory may be classified as "bounded" within the basin, yet still exhibit chaotic behavior with high divergence from neighboring trajectories. Quantifying this divergence is fundamental for practical applications, such as planning correction maneuvers or determining regions of greater predictability.

To address this gap, this work employs the Largest Lyapunov Exponent (LLE) as a quantitative metric of chaos. The LLE measures the average exponential rate of divergence between two initially close trajectories. A positive LLE is the definitive signature of deterministic chaos (SPROTT, 1997). Therefore, the main objective of this paper is to perform a quantitative analysis of chaos in the Earth-Moon system, correlating LLE maps with the escape basin structures for three distinct energy levels, determined by the Jacobi constant.

2 LITERATURE REVIEW AND METHODOLOGY

2.1 The Restricted Three-Body Problem (RTBP)

The RTBP describes the dynamics of a negligible mass particle (P_3) under the gravitational influence of two primary bodies (P_1) (Earth) and (P_2) (Moon) with masses (m_1) and (m_2). Considering a synodic (rotating) reference frame with origin at the system's barycenter, the system's dynamics are governed by a set of ordinary differential equations. The only known integral of motion is the Jacobi Constant (C), which is an energy-like quantity for the non-inertial reference frame. The equilibrium points in this reference frame are the five Lagrange Points $((L_1)$ to $(L_5))$, whose positions depend on the mass parameter $(\mu = m_2/(m_1 + m_2))$ (SZEBEHELY, 1967), which for the Earth-Moon system is $(\mu = 0.0121506683)$.

2.2 Escape Basins

The scattering region is defined between the collinear points (L_1) and (L_2) in the vicinity of the Moon. A set of initial conditions (x, y) with $(v_x = 0)$ and (v_y) determined by the Jacobi constant (C) is numerically integrated. Each trajectory is classified into one of the following categories (ASSIS; TERRA, 2014):

- **Collision with the Moon:** The distance to the primary (P_2) becomes smaller than the lunar radius (4.521e-3 in dimensionless units).
- **Earth's Domain:** The particle escapes through the (L_1) gateway and approaches the Earth.
- **Exterior Domain:** The particle escapes through the (L_2) gateway to outer space.
- **Confinement (Bounded):** The particle remains in the lunar region without colliding or escaping.

2.3 Largest Lyapunov Exponent (LLE)

The largest Lyapunov exponent quantifies the exponential separation rate between two neighboring orbits. For a dynamical system, it is defined as (SPROTT, 1997):

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{d(t)}{d_0}$$

where (d_0) is a small initial separation between two orbits in phase space, and $(d(t))$ is the separation after time (t).

For computational calculation, the periodic renormalization method was implemented (BENETTIN et al., 1980). The algorithm is summarized as follows:

- **Initialization**: An initial condition (z_0) and a perturbed orbit $(z_0' = z_0 + \delta)$ are defined, with $(\left||\delta|\right| = d_0 = 10^{-8})$. A renormalization time interval $(\Delta t = 0.2)$ is set.

- **Evolution and Readjustment**: Both orbits are integrated over a step (Δt). The new separation (d_1) and the ratio $(\ln(d_1/d_0))$ are calculated. The perturbed orbit is then readjusted to the distance (d_0) along the separation vector.
- **Calculation**: The process is repeated for N steps. The first $(n_{\text{transient}} = 20)$ steps are discarded. The LLE is the average of the logarithmic ratios accumulated after the transiente.

This algorithm was implemented to generate LLE maps, where each point of the discretized grid (x, y) receives a (λ) value.

3 RESULTS AND DISCUSSION

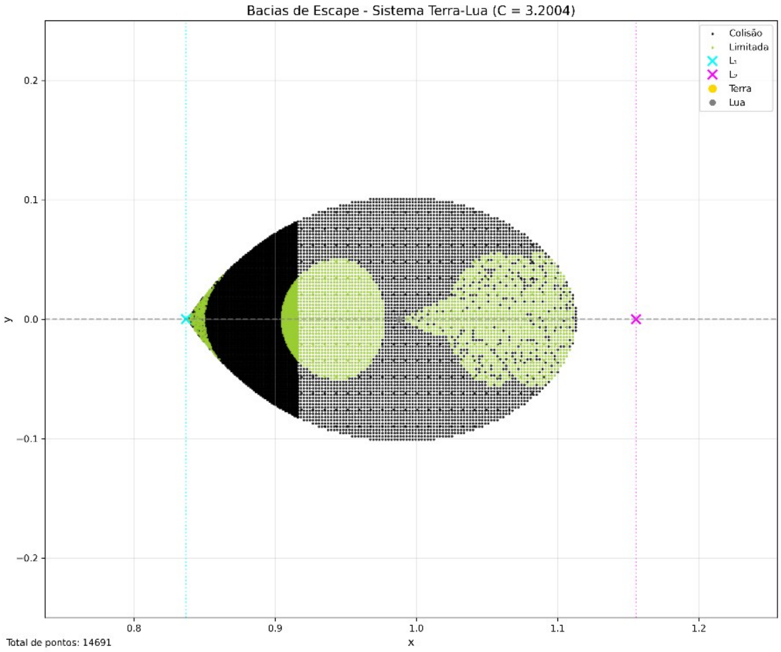
Escape basins and LLE maps were generated for three values of the Jacobi constant: $(C = 3.08)$ (high energy), $(C = 3.15)$ (intermediate energy), and $(C = 3.2004)$ (low energy). The analyzed scattering region covers $(x \in [x_{L2}, x_{L1}])$ (between the Lagrange points) and $(y \in [-0.2, 0.2])$, with a resolution of 150x150 points.

3.1 Low Energy Regime (C = 3.2004)

In this regime, the Hill regions are more closed, restricting motion.

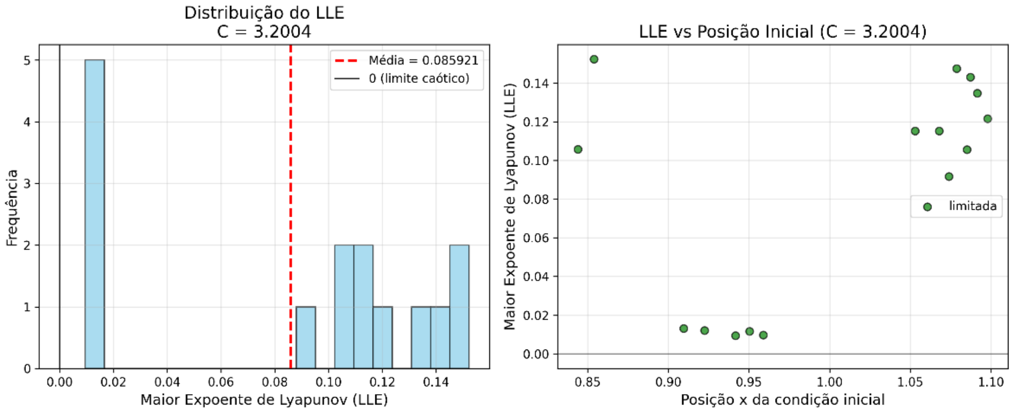
- **Escape Basin** (Figure 1): There is a predominance of the confinement basin (green), with islands of escape to the Earth's domain (black) and collision with the Moon (red). The structures are relatively well-defined.
- **LLE Map** (Figure 2): The map shows predominantly low LLE values, close to zero in the central region. LLE peaks occur in isolation, mainly at the boundaries of the small escape islands, indicating that instability is confined to these thin transition layers.

Figure 1 – Escape Basin for high C



Source: Authors' own elaboration.

Figure 2 – Lyapunov for high C



Source: Authors' own elaboration.

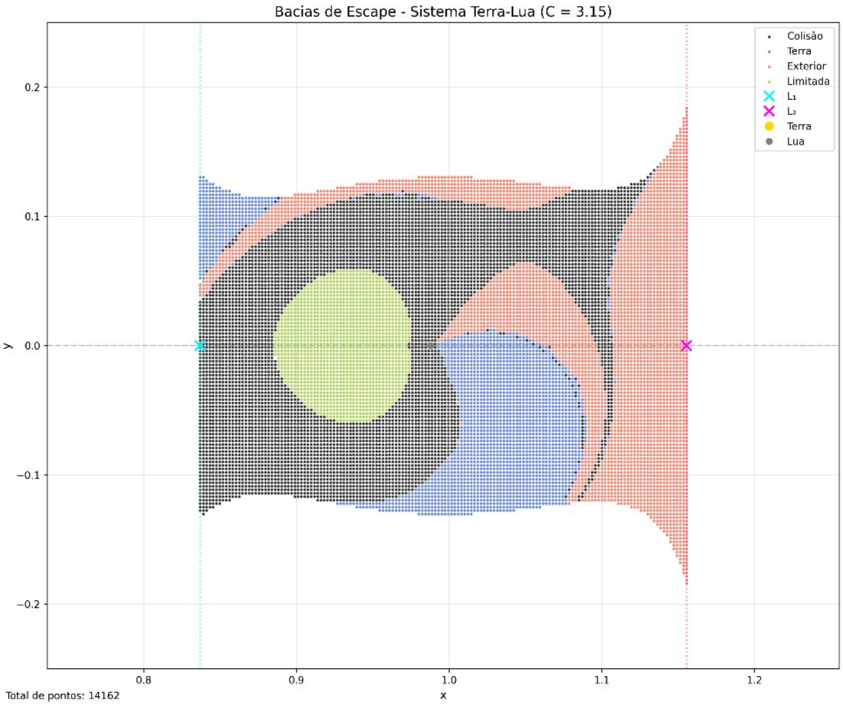
3.2 Intermediate Energy Regime (C = 3.15)

This is the transition regime, where the gateways at L_1 and L_2 are open.

- **Escape Basin** (Figure 3): The structural complexity is maximal. The collision, confinement, and escape to Earth basins intertwine in a rich fractal structure, with a high degree of mixing among the different categories.
- **LLE Map** (Figure 4): The LLE map exhibits an impressive correlation with the basin. The fractal boundary regions, where different basins meet, are precisely the regions with the highest LLE values. This quantitatively confirms the high unpredictability in these areas, since small variations in the initial condition lead

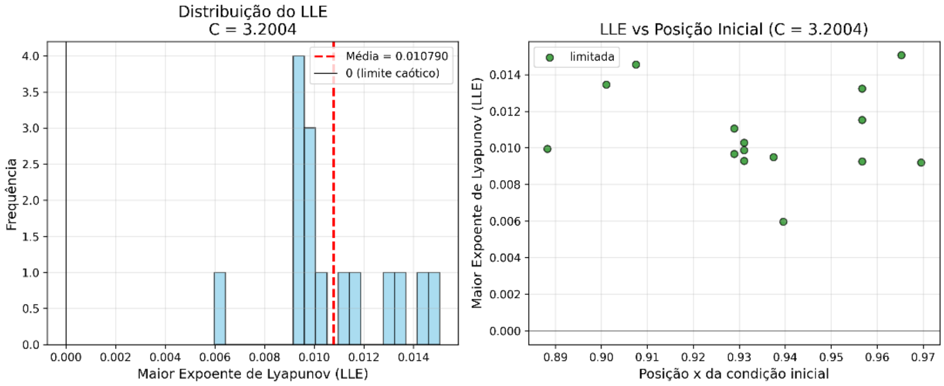
to drastically different final trajectories, reflected in a rapid exponential divergence represented by the high LLE.

Figure 3 – Escape Basin for intermediate C



Source: Authors' own elaboration.

Figure 4 – Lyapunov for intermediate C



Source: Authors' own elaboration.

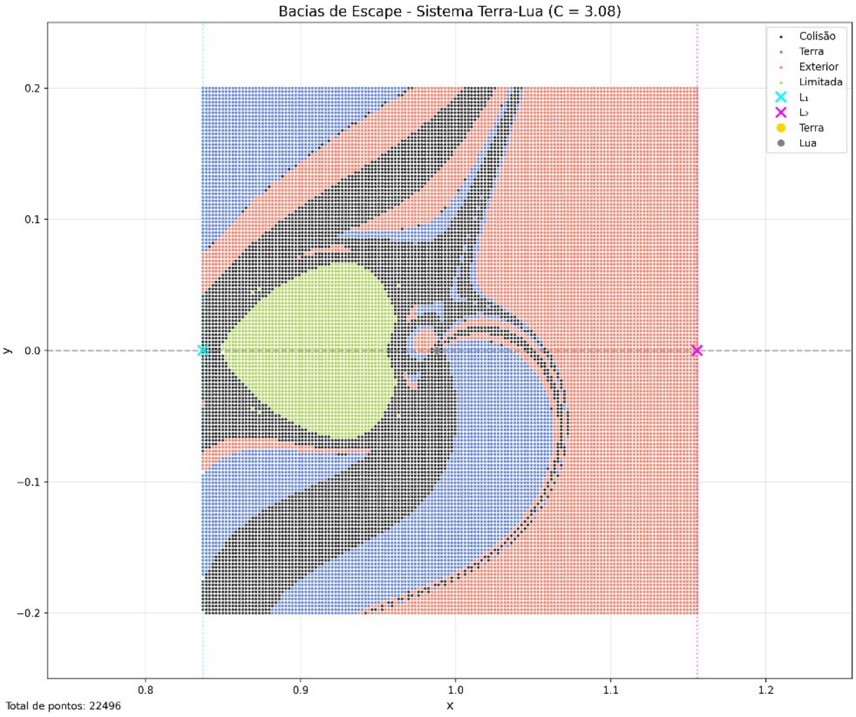
3.3 High Energy Regime (C = 3.08)

In this regime, the Hill regions are widely open, facilitating escape to the exterior domain.

- **Escape Basin** (Figure 3a): The basin is dominated by escape to the exterior domain (blue). The other basins are less prominent.
- **LLE Map** (Figure 3b): The map shows consistently positive LLE values throughout the region, with peaks at the still-visible boundaries between the

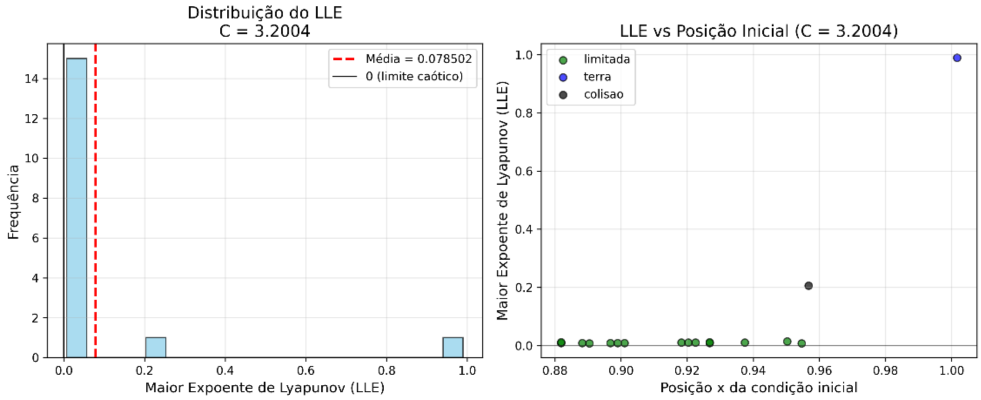
remaining basins. Although the final fate (exterior escape) is predictable, the chaotic path of the trajectory is evidenced by the positive LLE, indicating a high sensitivity to initial conditions during transit.

Figure 5 – Escape Basin for low C



Source: Authors' own elaboration.

Figure 6 – Lyapunov fro low C



Source: Authors' own elaboration.

4 FINAL REMARKS

A quantitative analysis of chaos in the Earth-Moon system was presented, correlating Largest Lyapunov Exponent (LLE) maps with the traditional escape basins of the Restricted Three-Body Problem. The results demonstrate that the LLE is a powerful and complementary tool, as it directly quantifies instability: regions qualitatively identified as chaotic, especially the fractal boundaries between basins, exhibit the highest LLE values, providing a precise numerical measure of the exponential divergence rate between neighboring trajectories.

Additionally, the LLE map reveals hidden dynamical structure that is not evident from the escape basin alone, as in the case of the high-energy regime, where widespread chaos is observed even when the final fate of the trajectories is the same (escape to the exterior domain).

Finally, the combination of both analyses directly aids in space mission design, as it allows identifying not only where a trajectory is heading, but also how sensitive and unpredictable it is. Regions with low LLE are ideal candidates for precision maneuvers or for establishing parking orbits, while regions with high LLE should be avoided or, when unavoidable, require active control systems to mitigate the effects of instability.

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